Limits and Strengths of Predicate Logic (and its Alloy Implementation)

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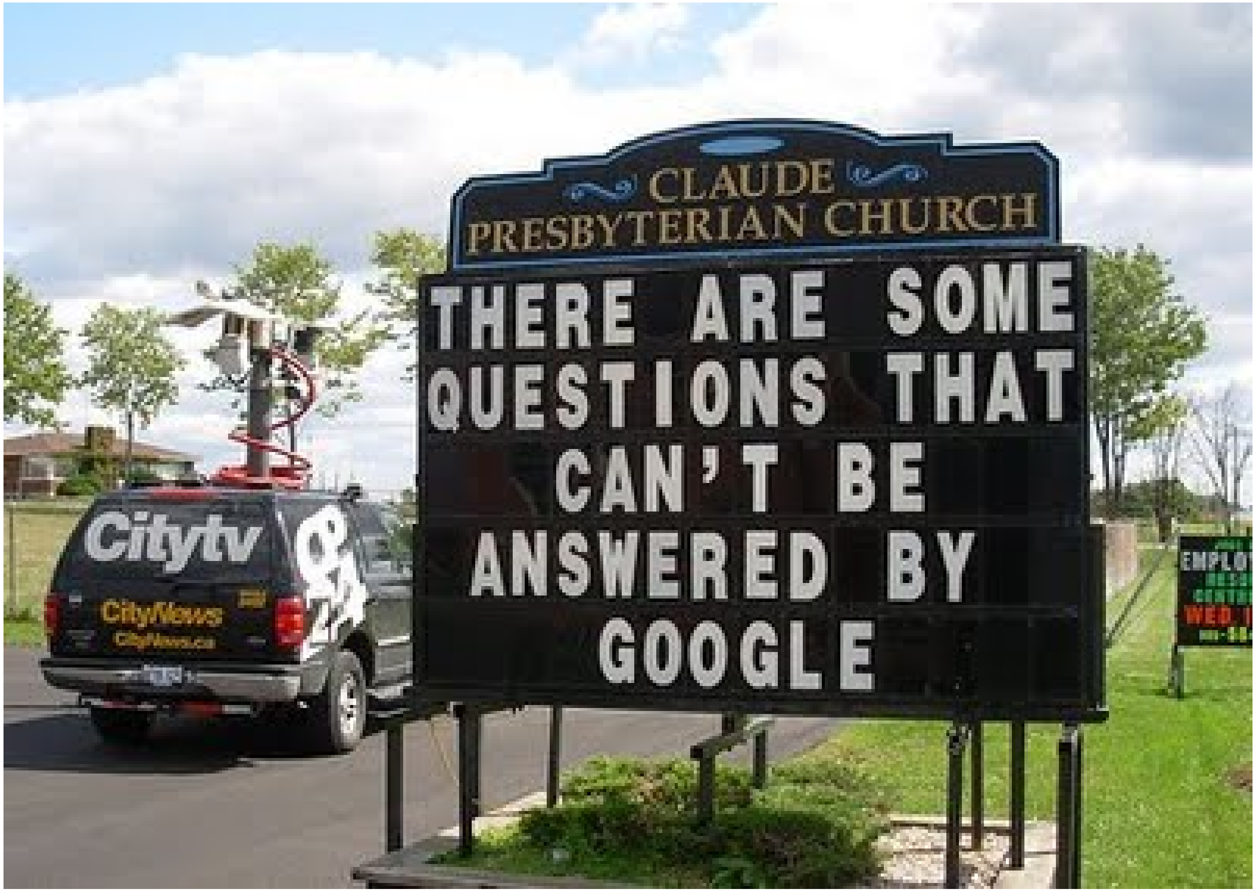
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There are some questions
that can't be answered by logic

There are some questions
that can't be answered
by computing machines

Automating First Order Relational Logic

- Relational logic = First Order Logic plus Relational Operators.
- Most relational operations are expressible in first order logic, but not all of them.
- Relation composition and relation transpose can be expressed in first order logic:
- `r.s` can be expressed as $\{(x, y) \mid \exists z R(x, z) \wedge S(z, y)\}$.
- In Alloy:

  ```
  x->y in r.s iff some z | x->z in r and z->y in s
  ```

- `~R` can be expressed as $\{(x, y) \mid R(y, x)\}$.
- In Alloy:

  ```
  x->y in ~r iff y->x in r
  ```

Beyond First Order Logic: Transitive Closure

- Transitive closure and reflexive transitive closure cannot be expressed in first order logic.
- The transitive closure of R is the smallest relation S for which:
  - $R \subseteq S$,
  - S is transitive.
- To express `^r` one would need an 'infinite formula':

$$
\begin{aligned}
\{(x,y) \ \mid \ & R(x,y) \lor \exists z(R(x,z) \land R(z,y)) \\
& \lor \exists z, v(R(x,z) \land R(z,v) \land R(v,y)) \\
& \lor \exists z, v, w(R(x,z) \land R(z,v) \land R(v,w) \land R(w,y)) \\
& \lor \qquad \ldots \qquad \}
\end{aligned}
$$

Propositional Logic

- Propositional logic: logic of propositions.
- Example formulas:
  - p,
  - $p \vee q$,
  - $p \rightarrow p \vee q$,
  - $p \vee q \Leftrightarrow \neg(\neg p \wedge \neg q)$.
- Why is propositional logic decidable and first order logic undecidable?
- Let us first see how first order logic is different from propositional logic.

SAT

- The following questions about propositional formulas are equivalent:
  - F_1 implies F_2,
  - $F_1 \longrightarrow F_2$ is true for every valuation.
  - $F_1 \wedge \neg F_2$ is not satisfiable.
- The satisfiability problem for propositional logic is called SAT.

Decidability of Propositional Logic

- SAT is decidable
- Decision algorithm to check SAT of F:

  Any propositional formula mentions only a finite number of proposition letters. Say the proposition letters mentioned in F are $p_1, \ldots, p_n$.

  There are 2^n possible valuations for these letters.

  Just work out the truth value of F for each valuation (using the truth table method) to see if one of them is satisfiable.

  If this is the case, answer 'yes', otherwise answer 'no'.

The Truth Table Method

For instance, look at the formula

$$\neg p \wedge ((p \longrightarrow q) \Leftrightarrow \neg(q \wedge \neg p)).$$

Suppose p has value **1** and q has value **0**, then we get (using the truth tables):

- $\neg p$ has **0**,
- $p \longrightarrow q$ has **0**,
- $q \wedge \neg p$ has **0**;
- $\neg(q \wedge \neg p)$ has **1**;
- $(p \longrightarrow q) \Leftrightarrow \neg(q \wedge \neg p)$ has **0**,
- the whole expression has **0**.

$$\begin{array}{cccccccccccc}
\neg & p & \wedge & ((p & \rightarrow & q) & \Leftrightarrow & \neg & (q & \wedge & \neg & p)) \\
\vdots & \mathbf{1} & \vdots & \mathbf{1} & \vdots & \mathbf{0} & \vdots & \vdots & \mathbf{0} & \vdots & \vdots & \mathbf{1} \\
\mathbf{0} & & \vdots & & \mathbf{0} & & \vdots & \vdots & & \vdots & \mathbf{0} & \\
 & & \vdots & & & & \vdots & \vdots & & \mathbf{0} & & \\
 & & \vdots & & & & \vdots & \mathbf{1} & & & & \\
 & & \vdots & & & & \mathbf{0} & & & & & \\
 & & \mathbf{0} & & & & & & & & &
\end{array}$$

In compressed form:

$$\begin{array}{cccccccccccc}
\neg & p & \wedge & ((p & \rightarrow & q) & \Leftrightarrow & \neg & (p & \wedge & \neg & p)) \\
\mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}
\end{array}$$

The method given above (the truth table method) can also be used as a decision algorithm for propositional consequence.

Assignment for this week: implement propositional equivalence checking in Haskell. (Needed for automated testing of conversion routines that are supposed to preserve logical equivalence.)

First Order Logic

- Now look at the case of first order logic.
- A truth table method does not work here.
- Instead of valuations we need models.
- A model has a domain, and interpretations for all predicates and relations.
- Look at $\forall x, y, z(Rxy \wedge Ryz \rightarrow Rxz)$
- Alloy version:

```
all x,y,z: Entity |
   x->y in r and y->z in r implies x->z in r
```

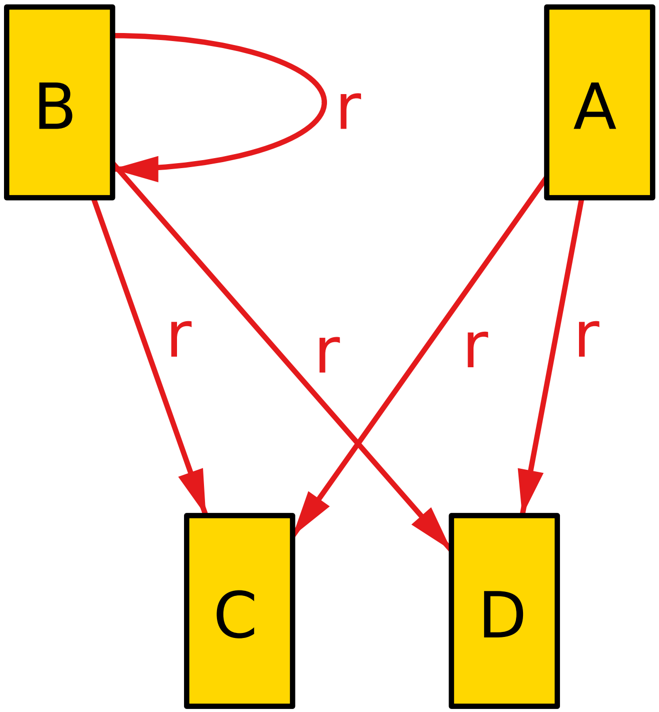
A model for this has a domain and a transitive relation on that domain. Let's try this out in Alloy . . .

Alloy Example

```
module myexamples/rel_t

abstract sig Entity { r: set Entity }

one sig A, B, C, D extends Entity {}

fact r_transitive {
 all x,y,z: Entity |
   x->y in r and y->z in r implies x->z in r }

pred show () {}
run show
```

Formulas and Models

- Consider the above Alloy specification.
- The result of executing **run show** for this specification is a model with a transitive relation r on it.
- Think of the Alloy specification as a formula.
- Think of the picture that results from executing **run show** as a model for that formula.


B
r
A
r
r
r
r
C
D

Another Way to Express Transitivity

```
module myexamples/rel_tt

abstract sig Entity { r: set Entity }

one sig A, B, C, D extends Entity {}

fact r_transitive {
 all x,y,z: Entity |
   x->y in r and y->z in r implies x->z in r }

assert r_transitive’ { r.r in r }
check r_transitive’
pred show () {}
run show
```

Formulas with Only Infinite Models

- Consider the conjunction of:
  - $\forall x \forall y (Rxy \rightarrow \neg Ryx) \quad (R \text{ is asymmetric})$
  - $\forall x \exists y Rxy \quad (R \text{ is serial})$
  - $\forall x \forall y \forall z ((Rxy \wedge Ryz) \rightarrow Rxz) \quad (R \text{ is transitive}).$
- Suppose our domain is non-empty.
- Then every model of this conjunction is infinite. Why?
- The task of checking all relational structures (including infinite ones) in search for a model of a formula cannot be finished in a finite amount of time.

Consistency, Refutation of Consistency

- A first order formula is consistent if it has a model.
- The existence for formulas with only infinite models suggests that first order consistency is not decidable.
- In fact, we have a semi-decision method: if a formula is inconsistent the method will determine this after finitely many steps.
- The method consists of constructing a so-called semantic tableau. This boils down to a systematic search for an inconsistency.
- There are consistent formulas for which the method loops. The refutation method for consistency is not an algorithm.
- Note that nothing we have said above is a proof that a decision method for first order consistency cannot exist.

Undecidable Queries

- The deep reason behind the undecidability of first order logic is the fact that its expressive power is so great that it is possible to state undecidable queries.
- One of the famous undecidable queries is the halting problem.
- Here is what a halting algorithm would look like:
  - Input: a specification of a computational procedure P, and an input I for that procedure
  - Output: an answer 'halt' if P halts when applied to I, and 'loop' otherwise.

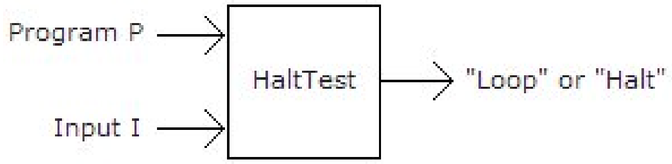
Undecidability of the Halting Problem

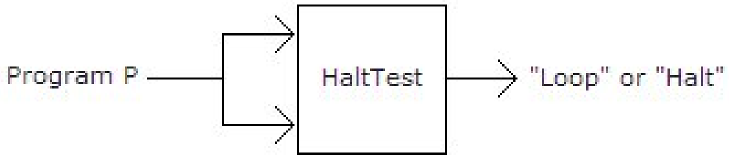
- Suppose there is an algorithm to solve the halting problem. Call this H.
- Then H takes a computational procedure P as input, together with an input I to that procedure, and decides. Note that H is itself a procedure; H takes two inputs, P and I.
- Let S be the procedure that is like H, except that it takes one input P, and then calls $H(P, P)$.
- Consider the following new procedure N for processing inputs P:
  If $S(P)$ says "halt", then loop,
  and if $S(P)$ says "loop", then print "halt" and terminate.

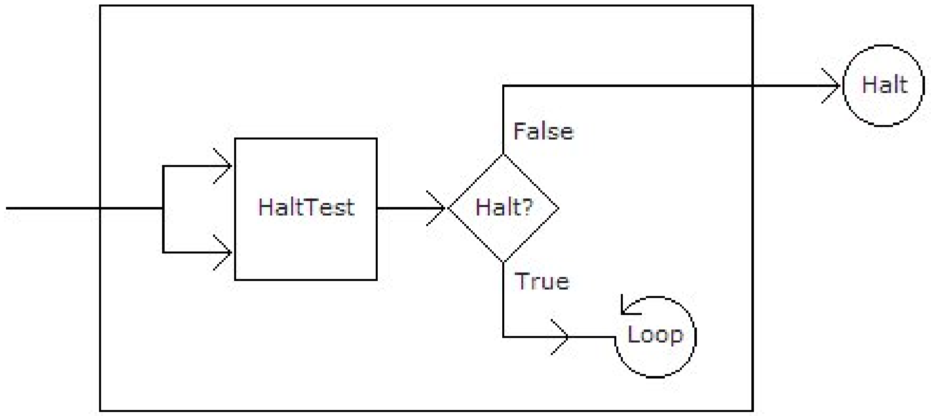
Undecidability of the Halting Problem (ctd)

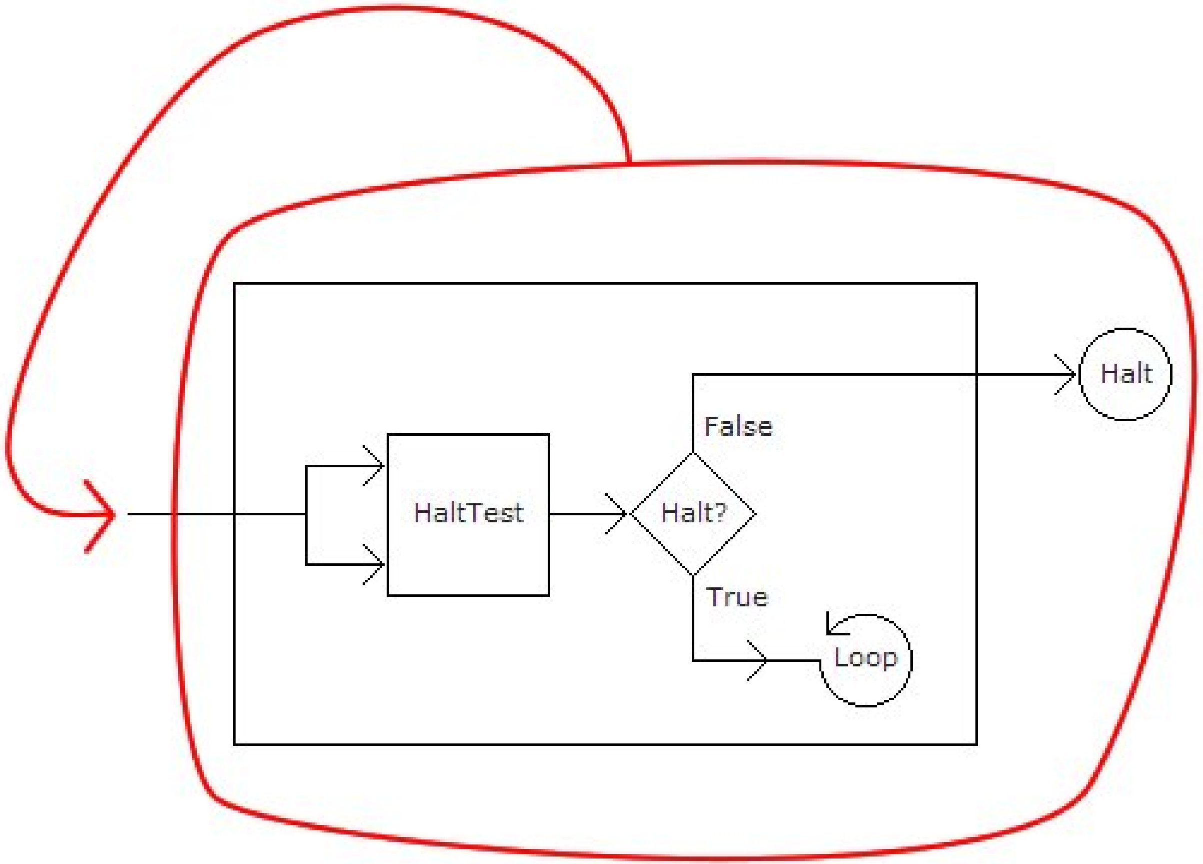
- What does N do when applied to N itself? In other words, what is the result of executing $N(N)$?
- Suppose N halts on input N. Then H should answer 'halt' when H is applied to N with input N, for H is supposed to be a correct halting algorithm. But then, by construction of the procedure, N loops. Contradiction.
- Suppose N loops on input N. Then H should answer 'loop' when H is applied to N with input N, for H is supposed to be a correct halting algorithm. But then, by construction of the procedure, N prints 'halt' and stops. Contradiction.
- We have a contradiction in both cases. Therefore a halting algorithm H cannot exist.

In Pictures . . .

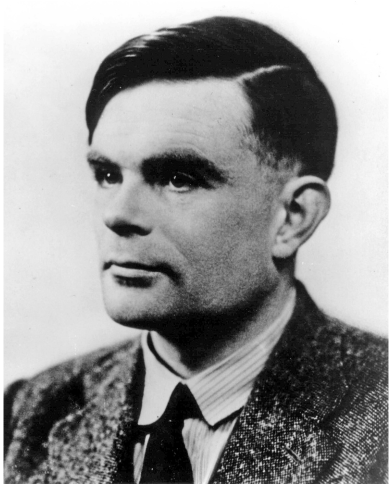

Program P
Input I
HaltTest
"Loop" or "Halt"


Program P
HaltTest
"Loop" or "Halt"


Halt
False
HaltTest
Halt?
True
Loop


HaltTest
Halt?
False
True
Halt
Loop

Alan Turing’s Insight

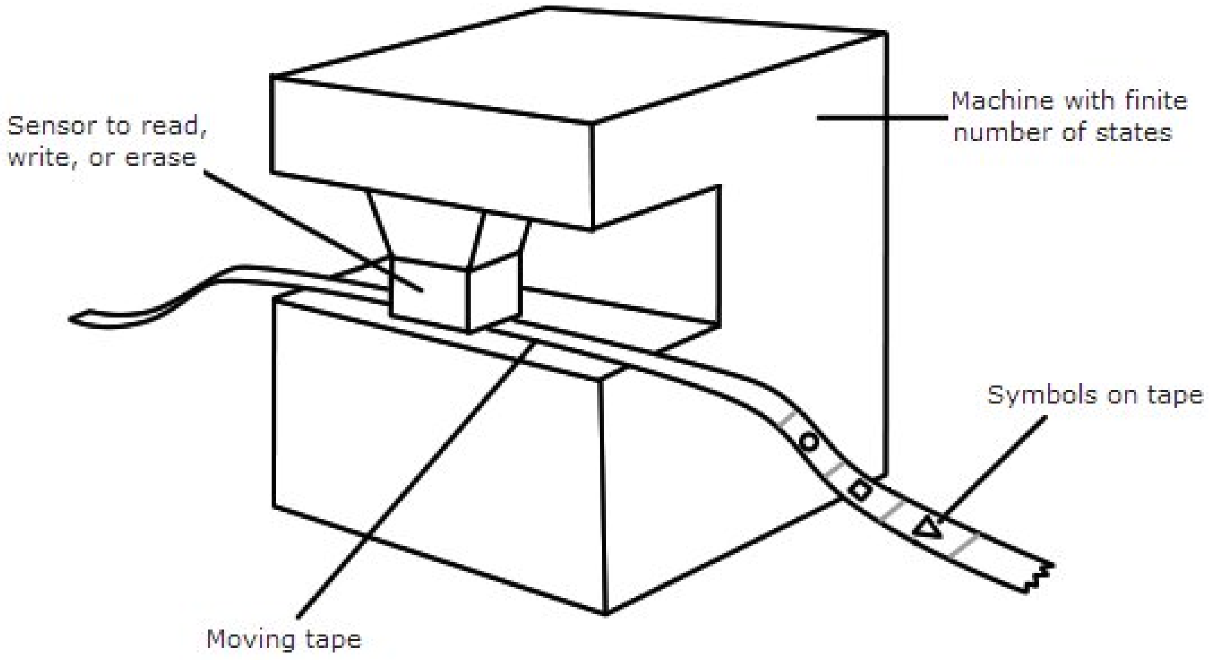


A language that allows the specification of ‘universal procedures’ such as H, S and N cannot be decidable.

But first order predicate logic is such a language …

Proof of Undecidability of First Order Logic

- The formal proof of the undecidability of first order logic consists of
  - A **very general** definition of computational procedures.
  - A demonstration of the fact that such computational procedures can be expressed in first order logic.
  - A demonstration of the fact that the halting problem for computational procedures is undecidable (see the above sketch).
  - A formulation of the halting problem in first order logic.
- This formal proof was provided by Alan Turing in **?**. The computational procedures he defined for this purpose were later called Turing machines.


Sensor to read, write, or erase
Machine with finite number of states
Symbols on tape
Moving tape

A Turing Machine

Turing Machines in Haskell

```
module Turing where

data State = Q | R | S | T | U | V | Stop
  deriving (Eq,Show)

data Symbol = Zero | One | Blank deriving Eq

instance Show Symbol where
  show Zero = "0"
  show One  = "1"
  show Blank = "*"

data Move = Lft State Symbol | Rght State Symbol | Halt
  deriving (Eq,Show)
```

```
type Tape = [Symbol]

type Instr = State -> Symbol -> Move

type Turing = (Tape,Int,State,Instr)

writeTape tape pos symbol =
  take pos tape  ++ [symbol] ++ drop (pos+1) tape

step :: Turing -> Turing
step (tape,h,state,f) =
   case
     f state (tape !! h) of
       Rght new symbol ->
         (writeTape tape h symbol,h+1,new,f)
```

```
      Lft  new symbol ->
        if h >= 0 then (writeTape tape h symbol,h-1,new,f)
        else (writeTape tape h symbol,h,new,f)
      Halt             ->  (tape,h,Stop,f)

compute :: Turing -> [(Tape,Int,State)]
compute (tape,h,Stop,f)    =  []
compute t@(tape,h,state,f) =
        (tape,h,state) : (compute . step) t

run :: Turing -> IO()
run turing = sequence_ (map print (compute turing))
```

Example Machines

```
module TuringExamples where

import Turing

turing1 :: Turing
turing1 =
  ([Blank,Blank, Zero,One,Zero,One,Blank,Blank],0,Q,f)
   where f Q Blank = Rght Q Blank
         f Q Zero  = Rght R One
         f Q One   = Rght R Zero
         f R Blank = Lft  S Blank
         f R Zero  = Rght R One
         f R One   = Rght R Zero
         f S Zero  = Lft  S Zero
         f S One   = Lft  S One
         f _   _   = Halt

turing2 :: Turing
turing2 = ([Blank,Zero,One,Zero,Zero,Blank],0,Q,f)
   where f Q Blank = Rght Q Blank
         f Q Zero  = Rght Q Zero
         f Q One   = Rght R One
         f R Zero  = Lft  S One
         f R One   = Rght R One
         f S Blank = Rght T Blank
         f S Zero  = Rght T Zero
         f S One   = Lft  S One
         f T Zero  = Rght Q Zero
         f T One   = Rght Q Zero
         f _   _    = Halt

turing2' :: Turing
turing2' =
  ([Blank,Zero,One,Zero,One,Zero,Zero,One,Blank],0,Q,f)
   where f Q Blank = Rght Q Blank
         f Q Zero  = Rght Q Zero
         f Q One   = Rght R One
         f R Zero  = Lft  S One
         f R One   = Rght R One
         f S Blank = Rght T Blank
         f S Zero  = Rght T Zero
         f S One   = Lft  S One
         f T Zero  = Rght Q Zero
         f T One   = Rght Q Zero
         f _   _   = Halt
```

Example Run

```
TuringExamples> run turing1
([*,*,0,1,0,1,*,*],0,Q)
([*,*,0,1,0,1,*,*],1,Q)
([*,*,0,1,0,1,*,*],2,Q)
([*,*,1,1,0,1,*,*],3,R)
([*,*,1,0,0,1,*,*],4,R)
([*,*,1,0,1,1,*,*],5,R)
([*,*,1,0,1,0,*,*],6,R)
([*,*,1,0,1,0,*,*],5,S)
([*,*,1,0,1,0,*,*],4,S)
([*,*,1,0,1,0,*,*],3,S)
([*,*,1,0,1,0,*,*],2,S)
([*,*,1,0,1,0,*,*],1,S)
```

Back to Alloy

- Alloy is not a decision method for first order logic
- Alloy translates first order logic with (small) finite scopes into propositional logic.
- Consistency of these translations can be decided, for propositional logic is decidable.
- SAT is intractable.
- More precisely, if one adds a single proposition letter, the computation time used by the truth table method doubles (for the truth tables become twice as big).
- This means that the truth table method is an exponential algorithm.

- It is very likely that all other methods for solving SAT are exponential, for SAT is NP-hard.

Sat and P=NP

- If we can solve SAT in polynomial time, we have solved the P = NP problem.
- P = NP is widely believed to be false, although nobody has been able to give a proof of this.
- What this means it that we should not expect the Alloy method to scale up.
- All future versions of Alloy will still only work for small domains.
- For a domain of size k, the result of adding an extra binary relation R to the signature is that 2^{k^2} possibilities for the interpretation of R have to be investigated.

Steps of the Alloy Analysis

1. Conversion to negation normal form and skolemization.
2. Translation (for a chosen scope) to a formula of propositional logic (a Boolean formula). Mapping between relational variables and Boolean variables is preserved.
3. Conversion of Boolean formula to conjunctive normal form.
4. CNF of Boolean formula fed to SAT solver.
5. If SAT solver finds a model, a first order version of this is constructed using the mapping from 2.

Checking Relational Properties with Alloy

- Asymmetry of a relation: $\forall x \forall y (Rxy \rightarrow \neg Ryx)$.
- Irreflexivity of a relation: $\forall x \neg Rxx$.
- Every asymmetric relation is irreflexive.

```
module myexamples/rel_asym

abstract sig Entity { r : set Entity }
one sig A, B, C, D extends Entity {}
fact r_asymmetric { no ~r & r }
assert r_irreflexive { no iden & r }
check r_irreflexive
pred show () { }
run show
```

Checking Relational Properties (ctd)

Do transitivity and symmetry together imply reflexivity?

```
module myexamples/rel_ts

abstract sig Entity { r : set Entity }
one sig A, B, C, D extends Entity {}

fact r_transitive { r.r in r }
fact r_symmetric { ~r in r }

pred show () { }
run show
assert r_reflexive { iden in r }
check r_reflexive
```

Checking Relational Properties (ctd)

Do transitivity, symmetry and seriality together imply reflexivity?

```
module myexamples/rel_tss
abstract sig Entity { r : set Entity }
one sig A, B, C, D extends Entity {}

fact r_transitive { r.r in r }
fact r_symmetric { ~r in r }
fact r_serial
 { all x: Entity | some y: Entity | x->y in r }

pred show () { }
run show
assert r_reflexive { iden in r }
check r_reflexive
```

Oops, Unexpected Alloy Behaviour

We get: counterexample found. Assertion is invalid.

This is strange, for the assertion is valid. If we check inspect counterexample, then we see what is in fact a reflexive relation. Very strange.

Debugging Alloy

Let's try a variation on the program:

```
module myexamples/rel_tss

abstract sig Entity { r : set Entity }

one sig A, B, C, D extends Entity {}

pred r_trans { r.r in r }
pred r_symm { ~r in r }
pred r_serial { all x: Entity | some y: Entity | x->y in r }
pred r_refl { iden in r }

TSSimpliesR: check {
   r_trans and r_symm and r_serial => r_refl
 }
```

Again, we get a report of a counterexample.

Again, if we inspect the ‘counterexample’, it turns out to be an example of a reflexive relation.

Debugging Alloy (ctd)

Let us do a further check. Select ‘show’ and open the Evaluator.

We get:

```
Eval> r

    {A$0->A$0, A$0->C$0, B$0->B$0, C$0->A$0, C$0->C$0,
    D$0->D$0}
```

This is our example of a reflexive relation. Let us check the iden relation:

```
Eval> iden

    {-8->-8, -7->-7, -6->-6, -5->-5, -4->-4, -3->-3,
    -2->-2, -1->-1, 0->0, 1->1, 2->2, 3->3, 4->4,
    5->5, 6->6, 7->7, A$0->A$0, B$0->B$0, C$0->C$0,
    D$0->D$0}
```

Oops, the identity is taken over a larger universe. Now it is no surprise that **`iden in r`** fails:

```
Eval> iden - r

    {-8->-8, -7->-7, -6->-6, -5->-5, -4->-4, -3->-3,
    -2->-2, -1->-1, 0->0, 1->1, 2->2, 3->3, 4->4,
    5->5, 6->6, 7->7}
```

Debugging Alloy (end)

OK, so the bug was in our definition of reflexivity. Now that we see this we can solve the problem:

```
module myexamples/rel_tss

abstract sig Entity { r : set Entity }

one sig A, B, C, D extends Entity {}

pred r_trans { r.r in r }
pred r_symm { ~r in r }
pred r_serial { all x: Entity | some y: Entity | x->y in r }
pred r_refl { iden in r }
pred r_refl' { all x: Entity |  x -> x in r }
```

```
TSSimpliesR: check {
   r_trans and r_symm and r_serial => r_refl
 }

TSSimpliesR’: check {
   r_trans and r_symm and r_serial => r_refl’
 }
```

If There is Time: Induction and Recursion